J. DIFFERENTIAL GEOMETRY
6 (1972) 595-598

THE SIGNATURE AND G-SIGNATURE OF MANIFOLDS WITH BOUNDARY

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1. The signature theorem

Let M be a compact oriented C^∞ manifold[1] of dimension $4k$ with boundary B of dimension $4k - 1$. The oriented B is called a *reflecting boundary* of M if it admits an orientation-reversing involution π. A simple example of the reflecting boundaries of M is a $(4k - 1)$-sphere. For convenience and simplicity, we shall always denote such as (B, π) a reflecting boundary with its involution together but with its dimension omitted. The following problem seems to be of interest: Is there any manifold M with a reflecting boundary (B, π) on which the involution π cannot be extended to the interior of the manifold M?

Now let $\tilde{M}$ with boundary $\tilde{B}$ be a C^∞ homeomorphic copy of (M, B) with the same orientation, and μ be the homeomorphism so that $\mu(M, B) = (\tilde{M}, \tilde{B})$. Then we can define the double of M with a reflecting boundary (B, π) to be a C^∞ closed oriented manifold N such that $N = M \cup \tilde{M}$ and that $\tilde{B} = \pi(B)$ by identifying $\mu\pi(x) \in \tilde{B}$ with x for all $x \in B$. Thus on the double N we can define a homeomorphism $\nu: N \to N$ by:

$$\nu(x) = \begin{cases} \mu(x) , & \text{for } x \in M , \\ \mu^{-1}(x) , & \text{for } x \in \tilde{M} . \end{cases} \tag{1.1}$$

To see that this is well-defined, at first we notice that $\nu(x) = \mu(x) \in \tilde{B}$ for $x \in \tilde{B}$. Since x is identified with $\mu\pi(x) \in \tilde{B}$, $\nu(x) = \nu(\mu\pi(x)) = \mu^{-1}(\mu\pi(x)) = \pi(x) \in B$, and therefore $\mu(x)$ is identified with $\pi(x)$; this is indeed true by the definition of our identification and the assumption $\pi^2 = 1$. Clearly, ν is an involution. (It should be noted that the definition of doubling a manifold M here is somewhat different from the ordinary one under which M and $\tilde{M}$ are of opposite orientations so that every point of B is a fixed point under the involution ν; for the latter see, for instance, [3].) Alternatively, we may regard doubling the manifold M with a reflecting boundary (B, π) as finding a C^∞ homeomorphism

$$h: M \to N$$

Communicated February 3, 1972, and, in revised form, June 25, 1972. Research partially supported by the National Science Foundation grant GP-33944.

[1] Throughout this paper all manifolds are differentiable.

where N is a C^∞ closed manifold with an involution

$$\nu: N \to N$$

such that $\nu h = h\pi: B \to N$. Assume h maps M into a fundamental domain of the involution ν in such a way that B is mapped onto itself. We shall identify M with the fundamental domain henceforth, so that we may regard the double N as composed of two halves M and $\tilde{M}$ with the same orientation such that M is mapped onto $\tilde{M}$ by ν, $M \cap \tilde{M} = B$, and $\nu | B = \pi$.

A Riemannian metric on the double N, for which the involution ν is an isometry, is said to be *symmetric*, and the restriction of a symmetric metric on N to M is also called a symmetric metric on M. Now there arises the problem of deriving a C^∞ symmetric Riemannian metric on N from a C^∞ Riemannian metric on M. For this problem at first we are naturally tempted to prolong to N a differentiable metric g on some manifold containing M by setting

$$g(x) = g(\nu(x)) \tag{1.2}$$

for $x \in M$. Although (1.2) is well defined, the difficulty is that the resulting metric will, in general, not be differentiable across B. However, on the other hand, for a given C^∞ Riemannian metric g everywhere defined on N we may obtain from it a C^∞ symmetric metric g by setting

$$g(x) = \tfrac{1}{2}[g(x) + g(\nu(x))] , \qquad \text{for } x \in N . \tag{1.3}$$

Now we consider a C^∞ symmetric Riemannian metric g on N. Then by Hirzebruch signature theorem [5] the signature of N is given by

$$\operatorname{sign}(N) = \int_N L_k(p_1, \cdots, p_k)(\Omega_g) , \tag{1.4}$$

where p_j is the j-th Pontrjagin class of N, and $\{L_k(p_1, \cdots, p_k)(\Omega_g)\}$ is the Hirzebruch's multiplicative sequence of polynomials with each p_i expressed in terms of the curvature 2-forms Ω_{jk} of the Riemannian metric g by a theorem of Chern [2]. However,

$$\begin{aligned} \int_N L_k(p_1, \cdots, p_k)(\Omega_g) &= \left(\int_M + \int_{\tilde{M}}\right) L_k(p_1, \cdots, p_k)(\Omega_g) \\ &= 2\int_M L_k(p_1, \cdots, p_k)(\Omega_g) , \end{aligned} \tag{1.5}$$

since $L_k(p_1, \cdots, p_k)(\Omega_g)$ depends only on the Riemmanian metric g, and the metric g is symmetric.

On the other hand, the following theorem was first observed by S. P. Novikov and proved jointly by Atiyah and Singer [1, Prop. (7.1)]:

Theorem 1.1. *Suppose that two compact oriented manifolds M_1 and M_2 have a common boundary B with opposite orientations. Then*

$$\operatorname{sign}(M_1 \cup M_2) = \operatorname{sign}(M_1, B) + \operatorname{sign}(M_2, B) , \tag{1.6}$$

where $\operatorname{sign}(M_i, B)$ *denotes the signature of the manifold M_i with boundary B for $i = 1, 2$.*

By applying Theorem 1.1 to our case we obtain

$$\operatorname{sign}(N) = \operatorname{sign}(M, B) + \operatorname{sign}(\tilde{M}, B) = 2 \operatorname{sign}(M, B) , \tag{1.7}$$

since M and $\tilde{M}$ are homeomorphic with the same orientation. Combination of (1.4), (1.5), (1.7) thus gives

$$\operatorname{sign}(M, B) = \int_M L_k(p_1, \cdots, p_k)(\Omega_g) . \tag{1.8}$$

Hence we arrive at

Theorem 1.2. *The signature of a compact oriented C^∞ manifold M of dimension $4k$ with a reflecting boundary B of dimension $4k - 1$ is given by* (1.8) *where g is a symmetric metric on M.*

When B is empty, Theorem 1.2 reduces to Hirzebruch signature theorem.

2. The G-signature and signature-defect

We first state the following theorem of Atiyah and Singer [1, p. 588], which is a generalization of Theorem 1.1 on the additivity property of the signature:

Theorem 2.1. *Suppose that two compact oriented manifolds M_1 and M_2 have a common boundary B with opposite orientations, and that a compact Lie group acts differentiably on M_1 and M_2 preserving the orientations. Then*

$$\operatorname{sign}(G; M_1 \cup M_2) = \operatorname{sign}(G; M_1, B) + \operatorname{sign}(G; M_2, B) , \tag{2.1}$$

where $\operatorname{sign}(G; M_1 \cup M_2)$ *and* $\operatorname{sign}(G; M_i, B)$ *denote, respectively, the G-signatures of $M_1 \cup M_2$ and M_i with boundary B for $i = 1, 2$.*

As in § 1 let M be a compact oriented manifold of dimension $2l$ with a reflecting boundary (B, π), and $N = M \cup \tilde{M}$ be the double of M with an involution ν defined by (1.1). Then we naturally intend to extend any automorphism g of M to an automorphism of N by

$$g(x) = \begin{cases} g(x) , & \text{for } x \in M , \\ \nu(g\nu^{-1}(x)) , & \text{for } x \in \tilde{M} , \end{cases} \tag{2.2}$$

so that

$$g(\nu\pi(x)) = \nu g \nu^{-1}(\nu\pi(x)) = \nu g \pi(x) \qquad \text{for } x \in B . \tag{2.3}$$

On the other hand, for $x \in B$ since $x = \nu\pi(x)$ by our identification for the reflecting boundary (B, π), we have

$$(2.4) \qquad g(x) = \nu\pi g(x) \; ,$$

which, together with (2.3), implies immediately that *to well define an automorphism of N by* (2.2) *it is necessary that*

$$(2.5) \qquad \pi g = g\pi \quad \text{on } B \; .$$

Now let M be a compact oriented manifold of dimension $2l$ without boundary, and suppose that there is a compact Lie group G acting on M preserving the orientation. For expressing sign $(g; M)$ for an element g of G, in [1] Atiyah and Singer obtained the G-signature Theorem (6.12), Corollary (6.13), Proposition (6.15), Corollary (6.16) (which was proved by Conner and Floyd [4, Cor. (27.4)] by a different method), and Proposition (6.18). If M has a reflecting boundary (B, π), by using our method of doubling a manifold in § 1 and Theorem 2.1, we can easily show that *the above mentioned expressions of Atiyah and Singer also hold for* sign $(g; M, B)$ *provided that on the boundary* B, g *has no fixed point and satisfies the condition* (2.5).

Very recently Hirzebruch [6] defined the signature-defect of a finite group acting effectively on a connected compact oriented manifold M without boundary, and obtained some interesting relationships between number theory and the signature-defect at some special points of a four-dimensional M.

Now let M be a connected compact oriented manifold of dimension 4 with a reflecting boundary (B, π), and suppose that there is a finite group G acting by orientation-preserving diffeomorphisms effectively on M and freely on B such that $\pi G = G\pi$ on B. By following Hirzebruch we can easily generalize his definition of the signature-defect of a group action on a manifold without boundary to the G-action on M, and show that *his relationships* [6, § 5] *between number theory and the signature-defect also hold for the signature-defect of the G-action on the manifold M with boundary* (B, π) by using the conditions of G on (B, π) and the extention in § 1 of Proposition (6.18) of Atiyah and Singer [1].

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